

ON THE RATIONALITY AND THE FINITE DIMENSIONALITY OF A CUBIC FOURFOLD

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ABSTRACT. Let X be a cubic fourfold in $\mathbf{P}_{\mathbf{C}}^5$. We prove that, assuming the Hodge conjecture for the product $S \times S$, where S is a complex surface, and the finite dimensionality of the Chow motive $h(S)$, there are at most a countable number of decomposable integral polarized Hodge structures, arising from the fibers of a family $f : S \rightarrow B$ of smooth projective surfaces. According to the results in [ABB] this is related to a conjecture proving the irrationality of a very general X . If X is special, in the sense of B.Hasset, and $F(X) \simeq S^{[2]}$, with S a K3 surface associated to X , then we show that the Chow motive $h(X)$ contains as a direct summand a "transcendental motive" $t(X)$ such that $t(X) \simeq t_2(S)(1)$. The motive of X is finite dimensional if and only if S has a finite dimensional motive, in which case $t(X)$ is indecomposable. Similarly, if X is very general and the motive $h(X)$ is finite dimensional, then $t(X)$ is indecomposable.

1. INTRODUCTION

Let $\mathcal{M}_{rat}(\mathbf{C})$ be the (covariant) category of Chow motives (with \mathbf{Q} -coefficients), whose objects are of the form (X, p, n) , where X is a smooth projective variety over \mathbf{C} of dimension d , p is an idempotent in the ring $A^d(X \times X) = CH^d(X \times X) \otimes \mathbf{Q}$ and $n \in \mathbf{Z}$. If X and Y are smooth projective varieties over \mathbf{C} , then the morphisms $\text{Hom}_{\mathcal{M}_{rat}}(h(X), h(Y))$ of their motives $h(X)$ and $h(Y)$ are given by correspondences in the Chow groups $A^*(X \times Y) = CH^*(X \times Y) \otimes \mathbf{Q}$. A similar definition holds for the category $\mathcal{M}_{hom}(\mathbf{C})$ of homological motives. The finite dimensionality of $h(X)$, as conjectured by S.Kimura, is known to hold for curves, rational surfaces, surfaces with $p_g(X) = 0$, which are not of general type and some 3-folds. In particular if X is a cubic 3-fold in $\mathbf{P}_{\mathbf{C}}^4$, then its motive is finite dimensional and of abelian type, i.e. it lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of abelian varieties, see[GG].

If $d = \dim X \leq 3$, then the finite dimensionality of $h(X)$ is a birational invariant (see[GG, Lemma 7.1]), the reason being that in order to make regular a birational map $\mathbf{P}^3 \rightarrow X$ one needs to blow up only

points and curves, whose motives are finite dimensional. Therefore every rational 3-fold has a finite dimensional motive.

In the case $d = \dim X = 4$ the situation looks different. The finite dimensionality of $h(X)$ is not a priori a birational invariant. In fact if $r : \mathbf{P}^4 \rightarrow X$ is a birational map then, by Hironaka theorem, there is a composition

$$(1.1) \quad \tau = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_1 : \tilde{X} \rightarrow \mathbf{P}^4$$

of monoidal transformations $\tau_i : \tilde{X}_i \rightarrow \tilde{X}_{i-1}$ with smooth centers C_{i-1} of dimensions ≤ 2 , where $\tilde{X}_0 = \mathbf{P}^4$ and $\tilde{X}_n = \tilde{X}$, such that $f = r \circ \tau : \tilde{X} \rightarrow X$ is a birational morphism. Therefore, by Manin's formula, we get

$$(1.2) \quad h(\tilde{X}) \simeq h(\mathbf{P}^4) \oplus \bigoplus_{1 \leq j \leq n-1} h(C_j)(j)$$

and hence $h(\tilde{X})$ is finite dimensional if all the motives $h(C_j)$ are finite dimensional, in which case also $h(X)$ is finite dimensional. Since the motives of points and curves are finite dimensional the problem is related to the finite dimensionality of those C_j such that $\dim C_j = 2$.

Let X be a cubic fourfold in $\mathbf{P}_{\mathbb{C}}^5$. All known examples are rational while the very general one is conjecturally irrational. These examples of rational cubic fourfolds have an associated K3 surface.

The aim of this note is to show how the rationality of a cubic fourfold and the finite dimensionality of its motive can be related to properties of the motives of surfaces.

In Sect.2 we recall some results by Y. Zarhin and V.S. Kulikov on cubic fourfolds. Zarhin in [Za] (see Theorem 2.1) showed that there is a restriction on the class of surfaces that can appear by resolving the indeterminacy of a rational map $r : \mathbf{P}^4 \rightarrow X$, as in (1.1). In particular, if one assumes that a very general cubic fourfold X is rational, then among the surfaces C_j appearing in (1.2) there must be surfaces S that are neither rational nor K3, see Remark 2.2. Kulikov in [Ku] (see Theorem 2.3) showed that a very general cubic fourfold in $\mathbf{P}_{\mathbb{C}}^5$ is not rational, if one assumes that the polarized Hodge structure \mathcal{T}_S on the transcendental cycles of a smooth projective surface is integrally indecomposable. However the above conjecture cannot be true because the polarized Hodge structure of a Fermat sextic $S \subset \mathbf{P}_{\mathbb{C}}^3$ is integrally decomposable, see [ABB].

In [ABB, Sect.4] it is also proved that the irrationality of a very general cubic fourfold X will follow, if one can prove that there are only countably many integral polarized Hodge structures, arising from the transcendental lattice of a surface, which are decomposable. There are

countably many families $\mathcal{S}_i \rightarrow B_i$ of complex surfaces, over an irreducible base variety, such that every isomorphism class of a projective surface is represented by a fiber of such a family. Therefore the irrationality of a very general cubic fourfold will follow if one could prove the following conjecture:

Conjecture 1.3. *Let $f : \mathcal{S} \rightarrow B$ be a smooth projective family of complex surfaces over an irreducible base. There are at most countably many integral polarized Hodge structures \mathcal{T}_{S_b} , arising from the fibers S_b of f , which are decomposable.*

In Sect. 3 we prove this conjecture for all families $f : \mathcal{S} \rightarrow B$ such that the generic fiber has an indecomposable transcendental motive, the closed fibers S_b have a finite dimensional motive and $S_b \times S_b$ satisfies the Hodge conjecture. We also compare our results with those appearing in [Gu] and in [ABB,4.7], see Remark 3.7.

In Sect 4 we consider the motive $h(X)$ of a cubic fourfold X such that the Fano variety of lines $F(X)$ is isomorphic to the Hilbert scheme $S^{[2]}$ of length-two subschemes of a K3 surface S associated to X . We prove (Theorem 4.6) that $h(X)$ contains as a direct summand a motive $t(X)$ which is isomorphic to $t_2(S)(1)$, where $t_2(S)$ is the transcendental motive of S . Examples of this type are given in 4.11(2) and 4.11 (3). As a corollary we get that $h(X)$ is finite dimensional if and only if $h(S)$ is finite dimensional (see Cor.4.9), in which case the motive $t(X)$ is indecomposable. Using this result we show that there is a family of K3 surfaces S , which are double covers of \mathbf{P}^2 ramified along a sextic and are associated to a family \mathcal{F} of cubic fourfolds in \mathbf{P}^5 , such that the motives $h(S)$ are finite dimensional and of abelian type, see 4.11(1). Finally we prove (Proposition 4.12) that for a very general cubic fourfold X with a finite dimensional motive $h(X)$ the transcendental motive $t(X)$ is indecomposable.

2. THE THEOREMS OF ZARHIN AND KULIKOV

Let X be cubic fourfold. A smooth surface S is said to be disjoint from X if one of the following conditions holds, where $\rho(S) = \dim_{\mathbf{Q}}(NS(S)_{\mathbf{Q}})$, $\rho_2(X) = \dim_{\mathbf{Q}} \bar{A}^2(X)$, with $\bar{A}^2(X) = \text{Im}(cl : A^2(X) \rightarrow H^4(X, \mathbf{Q}))$.

- (i) $h^{2,0}(S) = 0$;
- (ii) $h^{1,1}(S) - \rho(S) < 21 - \rho_2(X)$;
- (iii) $h^{2,0}(S) = 1$ and $h^{1,1}(S) - \rho(S) \neq 21 - \rho_2(X)$.

The following result by Y.Zarhin gives conditions for the class of surfaces that can appear when resolving the indeterminacy of a rational map $r : \mathbf{P}^4 \rightarrow X$.

Theorem 2.1. (*Zarhin*) *Let X be a cubic fourfold in $\mathbf{P}_{\mathbf{C}}^5$ and let $r : \mathbf{P}^4 \rightarrow X$ be a rational map of finite degree. It is not possible to make r regular by blowing up only points, curves, rational surfaces and surfaces S disjoint from X .*

Remark 2.2. If the cubic fourfold X is very general then any algebraic surface $S \subset X$ is homologous to a complete intersection, and hence $\rho_2(X) = 1$. Therefore the family of surfaces disjoint from X contains all K3 surfaces $S \subset X$, because $h^{2,0}(S) = 1$, $h^{1,1}(S) = 20$, $\rho(S) \geq 1$ so that

$$h^{1,1}(S) - \rho(S) \leq 19 < 20 = 21 - \rho_2(X)$$

The following example shows that, on the contrary, when X is not general and rational, then the locus of indeterminacy of a rational map $r : \mathbf{P}^4 \rightarrow X$ can consist of just one K3 surface S which is not disjoint from X .

Let $X \subset \mathbf{P}_{\mathbf{C}}^5$ be a cubic fourfold containing two disjoint planes P_1 and P_2 . Choose equations for P_1 and P_2 as follows

$$P_1 = \{u = v = w = 0\}, P_2 = \{x = y = z = 0\} \subset \mathbf{P}^5$$

where $\{u, v, w, x, y, z\}$ are homogeneous coordinates in \mathbf{P}^5 . Fix forms $F_1, F_2 \in \mathbf{C}[u, v, w, x, y, z]$ of bidegree $(1, 2)$ and $(2, 1)$ in the variables $\{u, v, w\}$ and $\{x, y, z\}$. Then the cubic hypersurface of equation $F_1 + F_2 = 0$ contains P_1 and P_2 . As in [Has 2, 1.2] one shows that cubic fourfolds of this type are rational. There is well defined morphism

$$\rho : (P_1 \times P_2 - S) \rightarrow X$$

which associates to every $(p_1, p_2) \in P_1 \times P_2$, such that the line $l(p_1, p_2)$ is not contained in X , the third intersection with X . The map ρ is birational because each point of $\mathbf{P}^5 - (P_1 \cup P_2)$ lies on a unique line joining the planes. Here S is the surface

$$S : \{F_1(u, v, w, x, y, z) = F_2(u, v, w, x, y, z) = 0\}.$$

which is a K3 surface, a complete intersection of bidegree $(1, 2)$ and $(2, 1)$, so that $\rho(S) = 2$. Choose an isomorphism $\Phi : P_1 \times P_2 \simeq \mathbf{P}^2 \times \mathbf{P}^2$, where $\{x, y, z\}$ are coordinates in the first copy and $\{u, v, w\}$ in the second one. Then we get a birational map $r : \mathbf{P}^4 \rightarrow X$ whose locus of indeterminacy is the K3 surface S , which is not disjoint from X . In fact we have $\rho_2(X) \geq 4$, because $\bar{A}^2(X)$ contains γ^2 , with γ the class of a hyperplane section, the classes of the two planes P_1 and P_2 and the class of S . Therefore $h^{1,1}(S) - \rho(S) = 18$, while $21 - \rho_2(X) \leq 17$.

A cubic fourfold X has Hodge numbers $h^{4,0} = h^{0,4} = 0$, $h^{3,1} = h^{1,3} = 1$ and $h^{2,2} = 21$. X satisfies the integral Hodge conjecture and hence

$$CH^2(X) = H^4(X, \mathbf{Z}) \cap H^{2,2}(X) \subset H^4(X, \mathbf{Z})$$

We also have $H^4(X, \mathbf{Z}) = \mathbf{Z}\gamma^2 \oplus P(X)_{\mathbf{Z}}$, where $\gamma \in H^2(X, \mathbf{Z}) = CH^1(X)$ is the class of a hyperplane section and $P(X)_{\mathbf{Z}}$ denotes the lattice of primitive cycles, i.e. $H^4(X, \mathbf{Z})_{\text{prim}} = P(X)_{\mathbf{Z}} = \ker(p : H^4(X) \rightarrow H^6(X))$, with $p(\alpha) = \alpha \cdot \gamma$. Poincaré' duality induces a symmetric bilinear form

$$\langle , \rangle : H^4(X) \times H^4(X) \rightarrow \mathbf{Z}$$

such that $\langle \gamma^2, \gamma^2 \rangle = 3$. The restriction of the bilinear form to $P(X)_{\mathbf{Z}}$ is non degenerate and hence yields a splitting

$$CH^2(X) \simeq \mathbf{Z}\gamma^2 \oplus CH^2(X)_0$$

with $CH^2(X)_0 = CH^2(X) \cap P(X)_{\mathbf{Z}}$.

The integral polarized Hodge structure on the primitive cohomology of X , is denoted by

$$\mathcal{H}_X = (P(X), H_{\text{prim}}^{p,q}, \langle , \rangle_{P(X)})$$

where $H_{\text{prim}}^{p,q} = H^{p,q}(X) \cap (H^4(X, \mathbf{Z})_{\text{prim}} \otimes_{\mathbf{Z}} \mathbf{C})$. Let $T_X \subset P(X)_{\mathbf{Z}} \subset H^4(X, \mathbf{Z})$ be the orthogonal complement of $CH^2(X)$ in $H^4(X, \mathbf{Z})$ with respect to \langle , \rangle , i.e. the lattice of transcendental cycles

$$H_{tr}^4(X, \mathbf{Z}) = T_X = \{h \in H^4(X) / \langle h, \alpha \rangle = 0, \alpha \in CH^2(X)\}.$$

If X is very general then $\rho_2(X) = 1$ and hence the rank of T_X is 22, because it is orthogonal, inside $H^4(X)$ of dimension 23, to $CH^2(X)$ which has dimension 1.

T_X has a natural polarized Hodge structure

$$\mathcal{T}_X = \{T_X, H_T^{p,q}, \langle , \rangle_T\}$$

with Hodge numbers $h^{3,1} = h^{1,3} = 1$, $h^{2,2} = 21 - \rho_2(X)$ and \langle , \rangle_{T_X} as a polarization. It is an integral polarized Hodge substructure of the primitive cohomology. For a very general X the polarized Hodge structure is irreducible.

A polarized Hodge structure \mathcal{T}_X is the direct sum of two integral polarized sub-Hodge structures $(T_{\mathbf{Z},1}, H_1^{p,q}, \langle , \rangle_{T_1})$ and $(T_{\mathbf{Z},2}, H_2^{p,q}, \langle , \rangle_{T_2})$ if

$$\mathcal{T}_X \simeq (T_{\mathbf{Z},1} \oplus T_{\mathbf{Z},2}, H_1^{p,q} \oplus H_2^{p,q}, \langle , \rangle_{T_1} \oplus \langle , \rangle_{T_2})$$

An integral polarized Hodge structure is indecomposable if it is not a direct sum of two non trivial integral polarized Hodge structures.

If S is a complex surface then the polarized Hodge structure \mathcal{T}_S on the transcendental cycles $H_{tr}^2(S) = T_S$ is given by

$$\mathcal{T}_S = (T_S, H_{T_S}^{p,q}, <, >_{T_S})$$

where $H_{T_S}^{p,q} = (T_S \otimes \mathbf{C}) \cap H^{p,q}(S)$ and $<, >_{T_S}$ is induced by the intersection form on $H^2(S, \mathbf{Z})$. The polarized Hodge structure \mathcal{T}_S is a birational invariant. If $p_g(S) \leq 1$ then \mathcal{T}_S is integrally indecomposable.

Let X be a rational cubic fourfold, $r : \mathbf{P}^4 \rightarrow X$ a birational map and let $f = r \circ \tau : \tilde{X} \rightarrow X$ be a birational morphism, as (1.1). Then we have a decomposition of the polarized Hodge structures

$$\mathcal{H}_X = \tau^*(\mathcal{H}_{\mathbf{P}^4}) \oplus \bigoplus_{0 \leq i \leq n-1} \mathcal{H}_i$$

where \mathcal{H}_i is the contribution in \mathcal{H}_X of the $(i+1)$ -th monoidal transformation τ_{i+1} .

V.S. Kulikov in [Ku, Lemma 2] proved the following result in the case X is a very general cubic fourfold.

Theorem 2.3. (*Kulikov*) *Let X be a very general smooth cubic fourfold which is rational and let $f = r \circ \tau : \tilde{X} \rightarrow X$ be as in (1.1). Then there is an index $i_0 \in \{1, \dots, n\}$ such that C_{i_0} is a smooth surface S and the polarized Hodge structure \mathcal{T}_S on the transcendental cycles $T_S = H_{tr}^2(S)$ can be decomposed into the direct sum of polarized Hodge structures*

$$\mathcal{T}_S = -f^*(\mathcal{T}_X)(1) \oplus \tilde{\mathcal{T}}_S$$

The above theorem is used in [Ku] to prove that a very general cubic fourfold is not rational, if one assumes the following conjecture: the integral polarized Hodge structure \mathcal{T}_S on the transcendental cycles on a smooth projective surface is non decomposable. However in [ABB] it is proved that the above conjecture cannot be true because the integral polarized Hodge structure of a Fermat sextic $S \subset \mathbf{P}_{\mathbf{C}}^3$ is decomposable. Note that, according to (2.1), among the surfaces C_1, \dots, C_n in Theorem 2.3 there are surfaces S which are not rational, not K3 and also not ρ -maximal, because if $h^{1,1}(S) = \rho(S)$, then S is disjoint from X .

3. HODGE CONJECTURE AND THE RATIONALITY OF A CUBIC FOURFOLD

In this section we prove that, assuming the Hodge conjecture for the product $S \times S$, where S is a complex surface, and the finite dimensionality of $h(S)$, there are at most a countable number of decomposable integral polarized Hodge structures in a family $f : \mathcal{S} \rightarrow B$ of smooth

projective surfaces, such that the generic fiber has an indecomposable transcendental motive

We first recall from [KMP, 7.2.2]) some results on the Chow-Künneth decomposition of the motive of surface.

Let S be a smooth projective surface over a field k . The motive $h(S) \in \mathcal{M}_{rat}(k)$ has a refined Chow-Künneth decomposition

$$h(S) = \sum_{0 \leq i \leq 4} h_i(S) = \sum_{0 \leq i \leq 4} (X, \pi_i)$$

where $h_2(S) = h_2^{alg}(S) + t_2(S)$, with $\pi_2 = \pi_2^{alg} + \pi_2^{tr}$, $t_2(S) = (S, \pi_2^{tr})$ and $h_2^{alg} = (S, \pi_2^{alg}) \simeq \mathbf{L}^{\oplus \rho(S)}$. Here $\rho = \rho(S)$ is the rank of $NS(S)$. The transcendental motive $t_2(S)$ is independent of the Chow-Künneth decomposition and is a birational invariant for S . We also have

$$H^i(t_2(S)) = 0 \text{ for } i \neq 2; \quad H^2(t_2(S)) = \pi_2^{tr} H^2(S, \mathbf{Q}) = H_{tr}^2(S, \mathbf{Q}),$$

$$A_i(t_2(S)) = \pi_2^{tr} A_i(S) = 0 \text{ for } i \neq 0; \quad A_0(t_2(S)) = T(S),$$

where $T(S)$ is the Albanese kernel. The map $\Phi : A^2(S \times S) \rightarrow \text{End}_{\mathcal{M}_{rat}}(t_2(S))$, defined by $\Phi(\Gamma) = \pi_2^{tr} \circ \Gamma \circ \pi_2^{tr}$, induces an isomorphism (see [KMP, 7.4.3])

$$(3.1) \quad \frac{A^2(S \times S)}{\mathcal{J}(S)} \simeq \text{End}_{\mathcal{M}_{rat}}(t_2(S))$$

where $\mathcal{J}(S)$ is the ideal in $A^2(S \times S)$ generated by correspondences whose support is contained either in $V \times S$ or in $S \times W$, with V, W proper closed subsets of S .

Lemma 3.2. *Let S be a smooth projective complex surface, with $p_g(S) \geq 2$, such that the polarized Hodge structure \mathcal{T}_S is integrally decomposable. Assume that*

- (i) *The variety $S \times S$ satisfies the Hodge conjecture;*
- (ii) *The motive $h(S)$ is finite dimensional.*

Then the transcendental motive $t_2(S)$ is decomposable.

Proof. Let $\mathcal{M}_{hom}(\mathbf{C})$ be the category of homological motives and let $PHS_{\mathbf{Q}}$ be the category of polarized Hodge structures, which is abelian and semisimple. There is a Hodge realization functor

$$H_{Hodge} : \mathcal{M}_{rat}(\mathbf{C}) \rightarrow PHS_{\mathbf{Q}}$$

which factors through $\mathcal{M}_{hom}(\mathbf{C})$. Let $h(S) = \sum_{0 \leq i \leq 4} h_i(S)$ be a refined Chow-Künneth decomposition with $h_2(S) = h_2^{alg}(S) + t_2(S)$. In the corresponding decomposition for the homological motive $h_{hom}(S)$ one has $h_{2,hom}(S) = t_2^{hom}(S) \oplus \mathbf{L}^{\oplus \rho(S)}$, where $t_2^{hom}(S) = (S, \pi_{2,hom}^{tr})$ is the image of $t_2(S)$ in \mathcal{M}_{hom} .

The Hodge structure \mathcal{T}_S , being decomposable, is the direct sum of two non-trivial polarized Hodge structures. Let $L \subset H_{tr}^2(S, \mathbf{Q})$ be a non-trivial Hodge substructure such that

$$H_{tr}^2(S, \mathbf{Q}) = L \oplus L'$$

where L' is the orthogonal complement of L with respect to the intersection pairing $Q = \langle \cdot, \cdot \rangle_{H_{tr}^2(S)}$. L defines a morphism of polarized Hodge structure

$$\Phi_L : H_{tr}^2(S, \mathbf{Q}) \rightarrow H_{tr}^2(S, \mathbf{Q})$$

such that $\ker \Phi = L'$ and $\text{Im } \Phi = L$. We also have $\Phi_L \circ \Phi_L = \Phi_L$. The Künneth decomposition and Poincaré' duality give isomorphisms

$$H^4(S \times S, \mathbf{Q}) = \bigoplus_{p+q=4} (H^p(S, \mathbf{Q}) \otimes H^q(S, \mathbf{Q})) \simeq \bigoplus_p \text{Hom}(H^p(S), H^p(S))$$

and these isomorphisms are isomorphisms of Hodge structures. The class $\Gamma_L \in H^4(S \times S, \mathbf{Q})$, which corresponds to Φ_L , is a Hodge class, because Φ_L is a morphism of Hodge structures, see [Vois 5, 1.4]. Therefore $\Gamma_L \in H^4(S \times S, \mathbf{Q}) \cap H^{2,2}(S \times S)$ and hence, by the Hodge conjecture, Γ_L is the class of an algebraic cycle, i.e. $\Gamma_L \in \bar{A}^2(S \times S)$, with $\bar{A}^2(S \times S) = \text{Im}(A^2(S \times S) \rightarrow H^4(S \times S, \mathbf{Q}))$. Then $\Gamma_L \in \text{End}_{\mathcal{M}_{hom}}(t_2^{hom}(S))$ and Γ_L is a projector different from 0 and from the identity. It follows that Γ_L defines a non-trivial submotive of $t_2^{hom}(S) = (S, \pi_{2,hom}^{tr})$. Therefore there exist a projector $\tilde{\pi}$ such that

$$\pi_{2,hom}^{tr} = \Gamma_L + \tilde{\pi}$$

with $\tilde{\pi} \neq \pi_{2,hom}^{tr}$. From the finite dimensionality of $h(S)$ we get that a similar decomposition also holds in $\mathcal{M}_{rat}(\mathbf{C})$ and hence

$$t_2(S) = (S, \pi_1) + (S, \pi_2)$$

where (S, π_i) is a non-trivial submotive of $t_2(S)$. Therefore $t_2(S)$ is decomposable in $\mathcal{M}_{rat}(\mathbf{C})$ \square

Example 3.3. (ρ -maximal surfaces) In [Beau] several examples are given of ρ -maximal surfaces S such that $h(S)$ is finite dimensional and $S \times S$ satisfies the Hodge conjecture. A complex surface S is ρ -maximal if $H^{1,1}(S) = NS(S) \otimes \mathbf{C}$ and hence the Picard number $\rho(S)$ equals $h^{1,1}(S)$. Then the subspace $H^{2,0} \oplus H^{0,2}$ of $H^2(S, \mathbf{C})$ is defined over \mathbf{Q} (see [Beau, Prop.1]) and it is the orthogonal complement of $H^{1,1}(S)$. Therefore $T_S \otimes \mathbf{C} = H^{2,0} \oplus H^{0,2}$ and hence the polarized Hodge structure \mathcal{T}_S decomposes over \mathbf{Q} . These examples include the Fermat sextic surface in \mathbf{P}^3 and the surface $S = C_6 \times C_6 \subset \mathbf{P}^4$, where C_6 is the plane sextic $x_0^6 + x_1^6 + x_2^6 = 0$. By Lemma 3.2 the motive $t_2(S)$ is decomposable. In the case of the Fermat sextic surface also the integral

polarized Hodge structure on the transcendental part $H_{tr}^2(S, \mathbf{Z})$ of the cohomology is decomposable, see [ABB, Theorem 1.2].

The following Lemma appears in [Gu, Lemma 13].

Lemma 3.4. *Let S be a smooth projective surface defined over a field k and let $k \subset L$ be a field extension. If the motive $t_2(S_L)$ is indecomposable then also $t_2(S)$ is indecomposable.*

Proof. Suppose that the motive $t_2(S)$ splits into two non-trivial direct summands T and T' in the category $\mathcal{M}_{rat}(k)$, with $T = (S, \pi)$ and $T' = (S, \pi')$. Then $\pi_2^{tr} = \pi + \pi'$, where π and π' are idempotents in $A^2(S \times S)$. Extending scalars from k to L we get a decomposition of $t_2(S_L)$ into the motives T_L and T'_L , by means of the projectors π_L and π'_L . Since the motive $t_2(S_L)$ is indecomposable we get that either π_L or π'_L is zero. Suppose $\pi_L = 0$. Then, by the results in [Gil], π_L is nilpotent and hence it vanishes. Therefore $T = 0$ which is a contradiction because T is a non-trivial summand of $t_2(S)$. \square

Theorem 3.5. *Let $f : \mathcal{S} \rightarrow B$ be a smooth projective family of complex surfaces with B an irreducible variety of dimension n . Let K be the algebraic closure of the function field of B and let S_K be the generic fibre of f over K . Assume that the transcendental motive $t_2(S_K)$ is indecomposable. Then there exists at most a countable number of $b \in B$ such that $t_2(S_b)$ is decomposable.*

Proof. By [Vial 4, Lemma 2.1] there exist a subset $U \subset B(\mathbf{C})$, which is a countable intersection of Zariski open subsets U_i , such that, for each point $b \in U$ there is an isomorphism between \mathbf{C} and K that yields an isomorphism between the scheme S_b over \mathbf{C} and the scheme S_K over K . Here $U_i = (B_0 - Z_i)(\mathbf{C})$ where B_0 is defined over a countable subfield $k \subset \mathbf{C}$ and Z_i is a proper subvariety of B_0 . Since the Chow groups of a variety X over a field only depend on X as a scheme we get an isomorphism

$$CH^2(S_b \times S_b) \simeq CH^2(S_K \times S_K)$$

which, by applying the isomorphism in (3.1), induces an isomorphism between the \mathbf{Q} -algebras $\text{End}_{\mathcal{M}_{rat}}(t_2(S_b))$ and $\text{End}_{\mathcal{M}_{rat}}(t_2(S_K))$. Since $t_2(S_K)$ is indecomposable $\text{End}_{\mathcal{M}_{rat}} t_2(S_K) \simeq \mathbf{Q}$. Therefore $\text{End}_{\mathcal{M}_{rat}}(t_2(S_b)) \simeq \mathbf{Q}$ and hence $t_2(S_b)$ is indecomposable, for all $b \in U(\mathbf{C}) = \bigcap_i U_i(\mathbf{C}) \subset B(\mathbf{C})$.

We claim that there exists at most a countable number of $b \in B(\mathbf{C})$ such that the corresponding fibre S_b has a decomposable transcendental motive.

If $t_2(S_b)$ is decomposable then $b \notin U(\mathbf{C})$ and hence there exists an index i_0 such that $b \in Z_{i_0}(\mathbf{C})$, with Z_{i_0} a subvariety of B_0 . Let $Z = Z_{i_0}$ with $\dim Z = m < n$. We proceed by induction on m .

If $m = 0$ then $Z(\mathbf{C})$ is just a finite collection of points in $B(\mathbf{C})$ and hence the corresponding fibers are a finite set. Assume that, if $Y \subset B_0$ with $\dim Y \leq m - 1$, there are only a countable number of $b \in Y(\mathbf{C})$ such that $t_2(S_b)$ is decomposable. Let $Z \subset B_0$ of dimension m and let $z \in Z(\mathbf{C})$ be such that $t_2(S_z)$ is decomposable. Since there are only a finite number of irreducible components of Z we may as well assume that Z is irreducible. Let η_Z be the generic point of Z and let $F = k(\eta_Z)$ be the function field of Z , which is a finitely generated extension of the countable subfield k of \mathbf{C} . Choose an embedding of the algebraic closure $L = \bar{F}$ into \mathbf{C} . Then $L \subset K$. By Lemma 3.4 $t_2(S_L)$ is indecomposable, because $t_2(S_K)$ is indecomposable. Therefore we may apply Lemma 2.1 in [Vial 4] to the family $f_Z : \mathcal{S}_Z \rightarrow Z$, where $\mathcal{S}_Z = \mathcal{S} \times_B Z$, whose generic fiber over the algebraic closure of $k(Z)$ is S_L . It follows that $t_2(S_z)$ is indecomposable for all points $z \in Z(\mathbf{C})$ in a countable intersection $V = \cap_j (Z - T_j)(\mathbf{C})$, where T_j is a proper subvariety of Z . Therefore, if $t_2(S_z)$ is decomposable, then there exists an index j_0 such that $z \in T_{j_0}(\mathbf{C})$. Since $\dim T_{j_0} \leq m - 1$, by the induction hypothesis, there are at most a countable number of points $z \in T_{j_0}(\mathbf{C}) \subset B(\mathbf{C})$ such that $t_2(S_z)$ is decomposable. Therefore there are at most a countable number of points in $z \in Z(\mathbf{C})$ such that the corresponding fiber S_z has a decomposable transcendental motive. \square

Corollary 3.6. *Let $f : \mathcal{S} \rightarrow B$ as in Theorem 3.5 and assume furthermore that for all surfaces in the family the motive $h(S)$ is finite dimensional and $S \times S$ satisfies the Hodge conjecture. Then the integral polarized Hodge structure \mathcal{T}_{S_b} arising from the fibers S_b which are decomposable are countably many.*

Proof. Let $b \in B$ be such that the polarized Hodge structure \mathcal{T}_{S_b} is integrally decomposable. By Lemma 3.2 the transcendental motive $t_2(S_b)$ is decomposable. From Theorem 3.5 we get that b belongs to a countable subset of $B(\mathbf{C})$ \square

Remark 3.7. (Integral decomposability) In [Gu, Thm.22] it is proven, using the Theorem of Kulikov, that a very general cubic fourfold X is not rational if one assumes that, for every smooth projective surface S , the transcendental motive $t_2(S)$ is integrally indecomposable and finite dimensional. Since the motive $t_2(S)$ lives in the category of \mathcal{M}_{rat} with rational coefficients it is not a priori clear what it means to be integrally decomposable. According to the definition given in [Gu] the motive $t_2(S)$ decomposes integrally if the identity of the ring $R = CH^2(S \times$

$S)/\mathcal{J}(S)$ is the sum of two orthogonal non-torsion idempotents, i.e. if there exist projectors Π_1, Π_2 such that the class $[\Delta_S]$ in R decomposes as $[\Delta_S] = [\Pi_1] + [\Pi_2]$. After tensoring with \mathbf{Q} the ring $R \otimes \mathbf{Q}$ becomes isomorphic to $\text{End}_{\mathcal{M}_{rat}}(t_2(S))$, as in (3.1), and the image of $[\Delta_S]$ is the identity of $t_2(S) = (S, \pi_2^{tr})$. Therefore $\pi_2^{tr} = \Phi(\Pi_1) + \Phi(\Pi_2)$, with $\Phi : A^2(S \times S) \rightarrow \text{End}_{\mathcal{M}_{rat}}(t_2(S))$.

Let $S = C_6 \times C_6 \subset \mathbf{P}^4$, where C_6 is the Fermat sextic in $\mathbf{P}_{\mathbf{C}}^2$. The surface S is ρ -maximal and the transcendental motive $t_2(S)$ is finite dimensional and decomposable in \mathcal{M}_{rat} , see Ex.3.3. In [Gu, Sect 5] it is proven that $t_2(S)$ is not integrally decomposable.

In [ABB, 4.7] it is noted that the irrationality of a very general cubic fourfold will follow if every complex surface S , such that the integral polarized Hodge structure \mathcal{T}_S is decomposable, has maximal rank. According to Theorem 2.1, among the surfaces S_i , with $1 \leq i \leq n$, that can appear when resolving the indeterminacy of a rational map $r : \mathbf{P}^4 \rightarrow X$, for a cubic fourfold X , there are surfaces which are not ρ -maximal, because these are all disjoint from X . On the other hand, by Theorem 2.3, there is an index i_o such that $\mathcal{T}_{S_{i_o}}$ is decomposable.

4. THE MOTIVE OF CUBIC FOURFOLD

Let X be a complex Fano 3-fold. By the results in [GG] its motive $h(X)$ has the following Chow-Künneth decomposition

$$h(X) = \mathbf{1} \oplus \mathbf{L}^{\oplus b} \oplus N \oplus (\mathbf{L}^2)^{\oplus b} \oplus \mathbf{L}^3$$

where $b = \dim NS(X)_{\mathbf{Q}}$ and $N = h_1(J) \otimes \mathbf{L}$, with $J = J^2(X)$ the intermediate Jacobian of X . Therefore $h(X)$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of Abelian varieties. In particular the motive of cubic 3-fold in $\mathbf{P}_{\mathbf{C}}^4$ is finite dimensional.

The above result is based on the fact that all Chow groups of a cubic 3-fold are representable. This is not the case if X is a cubic fourfold in $\mathbf{P}_{\mathbf{C}}^5$, because not all the Chow groups of X are representable. Indeed if, for a smooth projective variety, all Chow groups are representable, then the Hodge numbers $h^{p,q}$ vanish, whenever $|p - q| > 1$ (see [Vial 2, Thm.4]) and this is not the case for a cubic fourfold X , because $h^{3,1}(X) = h^{1,3}(X) = 1$.

In this section we show that, for a cubic fourfold X such that its Fano variety $F(X)$ is isomorphic to $S^{[2]}$, with S a K3 surface, then $h(X)$ is finite dimensional if and only if $h(S)$ is finite dimensional. Note that, by a result of R.Laterveer in [Lat], if $h(X)$ is finite dimensional then also $h(F(X))$ is finite dimensional.

Every cubic fourfold X is rationally connected and hence $CH_0(X) \simeq \mathbf{Z}$. Rational, algebraic and homological equivalences all coincide for cycles of codimension 2 on X . Hence the cycle map $CH^2(X) \rightarrow H^4(X, \mathbf{Z})$ is injective and $A^2(X) = CH^2(X) \otimes \mathbf{Q}$ is a sub vector space of dimension $\rho_2(X)$ of $H^4(X, \mathbf{Q})$. By the results in [TZ] we have $A_1(X)_{hom} = A_1(X)_{alg}$. Moreover homological equivalence and numerical equivalence coincide for algebraic cycles on X , because the standard conjecture $D(X)$ holds true. Therefore $A_1(X)_{hom} = A_1(X)_{num}$.

A cubic fourfold X has no odd cohomology and $H^2(X, \mathbf{Q}) \simeq NS(X)_{\mathbf{Q}} \simeq A^1(X)$, because $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Let $\gamma \in A^1(X)$ be the class of a hyperplane section. Then $H^2(X, \mathbf{Q}) = A^1(X) \simeq \mathbf{Q}\gamma$ and $H^6(X, \mathbf{Q}) = \mathbf{Q}[\gamma^2/3]$. Here $\langle \gamma^2, \gamma^2 \rangle = \gamma^4 = 3$, where $\langle \cdot, \cdot \rangle$ is the intersection form on $H^4(X, \mathbf{Q})$.

Let $\pi_0 = [X \times P_0]$, $\pi_8 = [P_0 \times X]$, where P_0 is a closed point and $\pi_2 = (1/3)(\gamma^3 \times \gamma)$, $\pi_6 = \pi_2^t = (1/3)(\gamma \times \gamma^3)$. Then

$$h(X) \simeq \mathbf{1} \oplus h_2(X) \oplus h_4(X) \oplus h_6(X) \oplus \mathbf{L}^4$$

where $\mathbf{1} \simeq (X, \pi_0)$, $\mathbf{L}^4 \simeq (X, \pi_8)$, $h_2(X) = (X, \pi_2)$, $h_6(X) = (X, \pi_6)$ and $h_4(X) = (X, \pi_4)$, with $\pi_4 = \Delta_X - \pi_0 - \pi_2 - \pi_6 - \pi_8$. The above decomposition of the motive $h(X)$ is in fact integral, because

$$\gamma^3 = 3|l|$$

for a line $l \in F(X)$, see [SV, Lemma A3].

Let ρ_2 be the dimension of $A^2(X)$ and let $\{D_1, D_2, \dots, D_{\rho_2}\}$ be a \mathbf{Q} -basis. We set

$$\pi_{4,i} = \frac{[D_i \times D_i]}{\langle D_i, D_i \rangle},$$

for $i = 1, \dots, \rho_2$. Then $\pi_{4,i}$ are idempotents and the motive $M_i = (X, \pi_{4,i}, 0)$ is isomorphic to the Lefschetz motive \mathbf{L} for every i . Let $h_4^{alg}(X)$ be the motive (X, π_4^{alg}) where $\pi_4^{alg} = \sum_{1 \leq i \leq \rho_2} \pi_{4,i}$. Then the motive $h_4(X)$ splits as follows

$$h_4(X) = h_4^{alg}(X) \oplus h_4^{tr}(X),$$

where $h_4^{alg}(X) \simeq \mathbf{L}^{\oplus \rho_2}$ and $h_4^{tr}(X) = (X, \pi_4^{tr})$, with $\pi_4^{tr} = \pi_4 - \pi_4^{alg}$.

Therefore we get the following Chow-Künneth decomposition of the motive $h(X)$

$$(4.1) \quad h(X) = \mathbf{1} + h_2(X) + \mathbf{L}^{\oplus \rho_2} + t(X) + h_6(X) + \mathbf{L}^4$$

where $t(X) = h_4^{tr}(X)$. The motives $h_2(X)$ and $h_6(X)$ are finite dimensional (see [Ki, 8.4.5]) and hence in the decomposition (4.1) all motives,

but possibly $t(X)$, are finite dimensional. Therefore the motive $h(X)$ is finite dimensional if and only if $t(X)$ is evenly finite dimensional.

Let $\mathcal{M}_{rat}^o(k)$ the category of birational motives (see [KMP, 7.5]). Writing $\bar{h}(X)$ for the birational motive of a smooth projective variety X of dimension d , and Hom_{bir} for morphisms in $\mathcal{M}_{rat}^o(k)$, the fundamental formula is:

$$A_0(X_{k(Y)}) = \varinjlim_{U \subset Y} A^d(U \times X) \cong \text{Hom}_{bir}(\bar{h}(Y), \bar{h}(X)).$$

If X is a cubic fourfold then $\bar{h}(X) = \mathbf{1}$ in $\mathcal{M}_{rat}^o(\mathbf{C})$, because X is unirational, see [Kahn, 7.3]. From the decomposition in (4.1) we get $\bar{t}(X) = 0$, because the Lefschetz motive \mathbf{L} goes to 0 in \mathcal{M}_{rat}^o . Therefore the identity map of $t(X)$ factors through a motive of the form $M(1)$, see [KMP, 7.5.3]. More precisely, by [Vial 3 2.2], the motive $t(X)$ is a direct summand of $h(Z)(1)$ for some surface Z and hence $t(X)$ is finite dimensional if $h(Z)$ is finite dimensional.

Lemma 4.2. *Let X be a cubic fourfold and let $t(X)$ be the transcendental motive in the Chow-Künneth decomposition (4.1). Then $A^i(t(X)) = 0$ for $i \neq 3$ and $A^3(t(X)) = A_1(X)_{hom}$*

Proof. The cubic fourfold X is rationally connected and hence $A^4(X) = A_0(X) = 0$ that implies $A^4(t(X)) = 0$. Also from the Chow-Künneth decomposition in (4.1) we get $A^0(t(X)) = 0$ and $A^1(t(X)) = \pi_4^{tr} A^1(X) = 0$, because $A^1(h_2(X)) = \pi_2 A^1(X) = A^1(X)$

We first show that $A^2(t(X)) = 0$. Let $\alpha \in A^2(X)$. Then

$$\pi_4^{tr}(\alpha) = \alpha - \pi_0(\alpha) - \pi_2(\alpha) - \pi_4^{alg}(\alpha) - \pi_6(\alpha) - \pi_8(\alpha),$$

where $\pi_0(\alpha) = \pi_8(\alpha) = 0$. We also have $\pi_2(\alpha) = (1/3)(p_2)_*(p_1^*(\alpha) \cdot [\gamma^3 \times \gamma]) = 0$ and $\pi_6(\alpha) = (1/3)(p_2)_*(p_1^*(\alpha) \cdot [\gamma^1 \times \gamma^3]) = 0$. Let $\alpha \in A^2(X)$ and let $\pi_4^{alg} = \sum_{1 \leq i \leq \rho_2} \pi_{4,i}$, with $\pi_{4,i} = [D_i \times D_i] / \langle D_i, D_i \rangle$, where $\{D_1, \dots, D_{\rho_2}\}$ is a \mathbf{Q} -basis for $A^2(X)$. Let $\alpha = \sum_{1 \leq i \leq \rho_2} m_i D_i$, with $m_i \in \mathbf{Q}$. Then $\pi_4^{alg}(\alpha) = \alpha$, because $(\pi_{4,i})_*(D_i) = D_i$. From the decomposition in (4.1) we get $\pi_4^{tr}(\alpha) = \alpha - \pi_4^{alg}(\alpha) = 0$ and hence

$$A^2(t(X)) = (\pi_4^{tr})_* A^2(X) = 0.$$

Therefore we are left to show that $A_1(t(X)) = A_1(X)_{hom}$. Let $\beta \in A_1(X) = A^3(X)$. From the Chow-Künneth decomposition in (4.1) we get

$$\pi_4^{tr}(\beta) = \beta - \pi_0(\beta) - \pi_2(\beta) - \pi_4^{alg}(\beta) - \pi_6(\beta) - \pi_8(\beta),$$

where $\pi_0(\beta) = \pi_4^{alg}(\beta) = \pi_8(\beta) = 0$. We also have $\pi_2(\beta) = 0$ because $\pi_2 = (1/3)(\gamma^3 \times \gamma)$. Therefore

$$\pi_4^{tr}(\beta) = \beta - \pi_6(\beta) = \beta - (1/3)(\gamma \times \gamma^3)_*(\beta) = \beta - (1/3)(\beta \cdot \gamma) \gamma^3 \in A^3(X)$$

and hence

$$(\pi_4^{tr}(\beta) \cdot \gamma) = (\beta \cdot \gamma - (1/3)(\beta \cdot \gamma)\gamma^3) \cdot \gamma = 0$$

because $\gamma^4 = 3$. Since γ is a generator of $A^1(X)$ it follows that the cycle $\pi_4^{tr}(\beta)$ is numerically trivial. Therefore we get

$$A_1(t(X)) = \pi_4^{tr} A_1(X) = A_1(X)_{num} = A_1(X)_{hom}$$

.

□

The following Lemma comes from the results in [Has 3, Thm. 3.18] and [GG, Lemma 1].

Lemma 4.3. *Let $f : M \rightarrow N$ be a morphism of motives in $\mathcal{M}_{rat}(\mathbf{C})$ such that $f_* : A^i(M) \rightarrow A^i(N)$ is an isomorphism for all $i \geq 0$. Then f is an isomorphism.*

Proof. Let $M = (X, p, m)$ and $N = (Y, q, n)$ and let $k \subset \mathbf{C}$ be a field of definition of f , which is finitely generated. Then $\Omega = \mathbf{C}$ is a universal domain over k . By [Has 3, Thm. 3.18] the map f has a right inverse, because the map $f_* : A^i(M) \rightarrow A^i(N)$ is surjective. Let $g : N \rightarrow M$ be such that $f \circ g = id_N$. Then g has an image T which is a direct factor of M and hence f induces an isomorphism of motives in $\mathcal{M}_{rat}(\mathbf{C})$

$$f : M \simeq N \oplus T$$

From the isomorphism $A^1(M) \simeq A^1(N)$, for all $i \geq 0$, we get $A^i(T) = 0$ and hence $T = 0$, by [GG, Lemma1]. □

Let X be a cubic fourfold and let $F(X) = F$ be its Fano variety of lines, which is a smooth fourfold. Let

$$(4.4) \quad \begin{array}{ccc} P & \xrightarrow{q} & X \\ p \downarrow & & \\ & & F \end{array}$$

be the incidence diagram, where $P \subset X \times F$ is the universal line over X . let $p_* q^* : H^4(X, \mathbf{Q}) \rightarrow H^2(F, \mathbf{Q})$ be the Abel-Jacobi map. Let $\alpha_1, \dots, \alpha_{23}$ be a \mathbf{Q} -basis of the vector space $H^4(X, \mathbf{Q})$. Then, by a result of Beauville -Donagi, $\tilde{\alpha}_i = p_* q^*(\alpha_i)$ form a basis of $H^2(F, \mathbf{Q})$. Let q_F be the Beauville-Bogomolov bilinear form on $H^2(F, \mathbf{Q})$ defined as

$$q_F(\tilde{\alpha}_i, \tilde{\alpha}_j) = \langle \alpha_i, \gamma^2 \rangle \cdot \langle \alpha_j, \gamma^2 \rangle - \langle \alpha_i, \alpha_j \rangle.$$

where \langle, \rangle is the symmetric bilinear form on $H^4(X, \mathbf{Q})$. Suppose that $F \simeq S^{[2]}$, with S a K3 surface. Then the homomorphism $H^2(S, \mathbf{Q}) \rightarrow$

$H^2(F, \mathbf{Q})$ induces an orthogonal direct sum decomposition with respect to the Beauville-Bogomolov form

$$(4.5) \quad H^2(F, \mathbf{Q}) \simeq H^2(S, \mathbf{Q}) \oplus \mathbf{Q}\delta,$$

with $q_F(\delta, \delta) = -2$ and q_F restricted to $H^2(S, \mathbf{Q})$ is the intersection form, see [SV,(56)].

A cubic fourfold X is special if contains a surface Σ such that its cohomological class σ in $H^4(X, \mathbf{Q})$ is not homologous to any multiple of γ^2 . Therefore $\rho_2(X) > 1$. The discriminant d is defined as the discriminant of the intersection form \langle, \rangle_D on the subspace D of $H^4(X, \mathbf{Q})$ generated by σ and γ^2 . By a result of B.Hassett in [Has 1, 1.0.3], if X is a generic special cubic fourfold with discriminant of the form $d = 2(n^2 + n + 1)$, where n is an integer ≥ 2 , then the Fano variety of X is isomorphic to $S^{[2]}$, with S a K3 surface associated to X . Such special X include cubic fourfolds containing a plane, or a cubic scroll, or a Veronese surface, see [Has 1, 4.1]. Special cubic fourfolds of discriminant d form a nonempty irreducible divisor \mathcal{C}_d in the moduli space \mathcal{C} of cubic four folds if and only if $d > 0$ and $d \equiv 0, 2(6)$. If d is not divisible by 4, 9 or any odd prime $p \equiv 2(3)$ then $X \in \mathcal{C}_d$ if and only if the transcendental lattice \mathcal{T}_X is Hodge isometric to $\mathcal{T}_S(-1)$ for some K3 surface, see [Add].

If X is special and $F(X) \simeq S^{[2]}$, with S a K3 surface, then by (4.5), $H_{tr}^2(F, \mathbf{Q}) \simeq H_{tr}^2(S, \mathbf{Q})$, where $\dim H_{tr}^2(F, \mathbf{Q}) = \dim H_{tr}^4(X, \mathbf{Q}) = 23 - \rho_2(X)$. Here $\rho_2(X) \geq 2$ and hence we get

$$\dim H_{tr}^2(F, \mathbf{Q}) = \dim H_{tr}^2(S, \mathbf{Q}) = 22 - \rho(S) = 23 - \rho_2(X) \leq 21,$$

where $\rho(S)$ is the rank of $NS(S)$.

Theorem 4.6. *Let X be a cubic fourfold and let $F = F(X)$ be the Fano variety of lines. Suppose that $F \simeq S^{[2]}$, with S a K3 surface. Let p and q be the morphisms in the incidence diagram (4.4). Then q induces a map of motives $\bar{q} : t_2(S)(1) \rightarrow t(X)$ in $\mathcal{M}_{rat}(\mathbf{C})$ which is an isomorphism.*

Proof. In (4.4) P is a \mathbf{P}^1 -bundle over F (see [Vois 4, 3.24]) and hence

$$h(P) \simeq h(F) \oplus h(F)(1)$$

By the results in [deC-M, 6.2.1] $h(S)(1)$ is a direct summand of $h(S^{[2]}) = h(F)$. Therefore we get a map of motives

$$(4.7) \quad \bar{q} : h(S)(1) \xrightarrow{f} h(F) \xrightarrow{g} h(P) \xrightarrow{q_*} h(X)$$

where f and g are the inclusions induced by the splittings of $h(F)$ and $h(P)$. Let $t_2(S)$ be the transcendental motive of the surface S . By

composing with the inclusion $t_2(S)(1) \rightarrow h(S)(1)$ and the surjection $h(X) \rightarrow t(X)$ we get from (4.7) a map of motives in $\mathcal{M}_{\text{rat}}(\mathbf{C})$,

$$\bar{q} : t_2(S)(1) \rightarrow t(X)$$

The morphisms p and q yield a homomorphism $\Psi_0 = p^*q_* : A^i(F) \rightarrow A^{i-1}(X)$ such that

$$\Psi_0 : A^4(F) = A_0(F) \rightarrow A^3(X) = A_1(X),$$

is surjective, see [TZ, 6.1(i)]. By [SV, 15.6] the group $A_0(F)$ decomposes as follows

$$A_0(F) = A_0(F)_0 + A_0(F)_2 + A_0(F)_4$$

where $A_0(F)_0 = \mathbf{Q}_{c_F}$, with $c_F \in A_0(F)$ a special degree 1 cycle. For two distinct points $x, y \in S$ let's denote by $[x, y] \in F = S^{[2]}$ the point of F that corresponds to the subscheme $\{x, y\} \subset S$. If $x = y$ then $[x, x]$ denotes the element in $A_0(F)$ represented by any point corresponding to a non reduced subscheme of length 2 on S supported on x . With these notations c_F is represented by the point $[c_S, c_S] \in F$, where c_S is the Beauville-Voisin cycle in $A_0(S)$ such that $c_2(S) = 24c_S$. We also have (see [SV, 15.6])

$$A_0(F)_2 = \langle [c_S, x] - [c_S, y] \rangle,$$

We claim that the map $\phi : A_0(S) \rightarrow A_0(S^{[2]}) = A_0(F)$ sending $[x]$ to $[c_S, x]$ is injective and hence $A_0(S)_0$ is isomorphic to $A_0(F)_2 \subset A_0(F)_{\text{hom}}$.

The variety $S^{[2]}$ is the blow-up of the symmetric product $S^{(2)}$ along S . Let \tilde{S} be the inverse image of S in $S^{[2]}$. Then \tilde{S} is the image of the closed embedding $s \rightarrow [c_S, s]$. By a result proved in [Ba] the induced map of 0-cycles $A_0(\tilde{S}) \rightarrow A_0(S^{[2]})$ is injective. Therefore the map ϕ is injective.

By [SV, 20.3] we have

$$A_0(F)_4 = A^2(F)_2 \cdot A^2(F)_2$$

where $A^2(F)_2 = \mathcal{A}_{\text{hom}}$ and \mathcal{A} is generated by the cycles of the form S_x with $x \in S$, see [SV, 15.2]. Here, for a point $x \in S$, the variety $S_x \subset F$ is isomorphic to the blow up of S at the point X and $S_x \cdot S_y = [x, y]$. Then, using the results in [SV, 20.2 (iii)] and [SV, 20.5], we get

$$\ker(\Psi_0 : A^4(F) \rightarrow A^3(X)) = F^4 A_0(F) = \mathcal{A}_{\text{hom}} \cdot \mathcal{A}_{\text{hom}} = A_0(F)_4.$$

Therefore Ψ_0 induces an exact sequence

$$A_0(F)_4 \rightarrow A_0(F)_{\text{hom}} \rightarrow A_1(X)_{\text{hom}} \rightarrow 0$$

which yields an isomorphism between $A_0(S)_0 = A_0(F)_2 \subset A_0(F)_{\text{hom}}$ and $A_1(X)_{\text{hom}}$.

We have $A^i(t_2(S)) = 0$ for $i \neq 2$ and $A^2(t_2(S)) = A_0(t_2(S)) = A_0(S)_0$. Therefore we get an isomorphism $A^3(t_2(S)(1)) = A^2(t_2(S)) = A_0(S)_{hom} \simeq A_1(X)_{hom}$. From Lemma 4.2 we get $A^i(t(X)) = 0$, for $i \neq 3$ and $A^3(t(X)) = A_1(X)_{hom}$.

Therefore the map of motives $\bar{q} : M \rightarrow N$ in $\mathcal{M}_{rat}(\mathbf{C})$, where $M = t_2(S)(1)$ and $N = t(X)$, induces isomorphisms $A^i(M) \simeq A^i(N)$ for all $i \geq 0$. Then, from Lemma 4.3, \bar{q} is an isomorphism. \square

Remark 4.8. Let X be a cubic fourfold such that there exist K3 surfaces S_1 and S_2 and isomorphisms $r_1 : F(X) \rightarrow S_1^{[2]}$ and $r_2 : F(X) \rightarrow S_2^{[2]}$ with $r_1^* \delta_1 \neq r_2^* \delta_2$, as in [Has 1, 6.2.1]. Then by Theorem 4.6 $t_2(S_1) \simeq t_2(S_2)$, and hence the motives $h(S_1)$ and $h(S_2)$ are isomorphic.

Corollary 4.9. *Let X be a cubic fourfold and let $F = F(X)$ be the Fano variety of lines. Suppose that $F \simeq S^{[2]}$, with S a K3 surface. Then $h(X)$ is finite dimensional if and only if $h(S)$ is finite dimensional in which case the motive $t(X)$ is indecomposable.*

Proof. If $h(X)$ is finite dimensional then also $t(X)$ is finite dimensional and hence, by Theorem 4.6, $t_2(S)$ is finite dimensional. Therefore $h(S)$ is finite dimensional. Conversely, if $h(S)$ is finite dimensional then also $t_2(S)$ and $t(X)$ are finite dimensional, by Theorem 4.6. From the Chow-Künneth decomposition in (4.1) we get that $h(X)$ is finite dimensional. If $h(S)$ is finite dimensional then the motive $t_2(S)$ is indecomposable, see [Vois 1, Cor. 3.10], and hence also $t(X)$ is indecomposable. \square

Remark 4.10. If the motive $h(X)$ is finite dimensional then $t(X)$ is, up to isomorphisms in $\mathcal{M}_{rat}(\mathbf{C})$, independent of the Chow-Künneth decomposition $h(X) = \sum_i h_i(X)$ in (4.1). If $h(X) = \sum_i \tilde{h}_i(X)$ is another Chow-Künneth decomposition, with $\tilde{h}_i(X) = (X, \tilde{\pi}_i)$, then, by [KMP, 7.6.9], there is an isomorphism $\tilde{h}_i(X) \simeq h_i(X)$ and $\tilde{\pi}_i = (1 + Z) \circ \pi_i \circ (1 + Z)^{-1}$, where $Z \in A^4(X \times X)_{hom}$ is a nilpotent correspondence. In particular

$$\tilde{\pi}_4 = (1 + Z) \circ \pi_4 \circ (1 + Z)^{-1} = (1 + Z) \circ (\pi_4^{alg} + \pi_4^{tr}) \circ (1 + Z)^{-1}$$

and hence $\tilde{h}_4(X)$ contains as a direct summand a submotive $\tilde{t}(X) = (X, (1 + Z) \circ \pi_4^{tr} \circ (1 + Z)^{-1})$ isomorphic to $t(X)$.

However, differently from the case of the transcendental motive $t_2(S)$ of a surface S , the motive $t(X)$ is not a birational invariant. In fact $t(X) \neq 0$ for a rational cubic fourfold X such that $F(X) \simeq S^{[2]}$, with S a K3 surface

Example 4.11. (1) A family \mathcal{F} of cubic fourfolds X with a finite dimensional motive is given by the equations

$$u^3 + v^3 + f(x, y, z, t) = 0$$

where (x, y, z, t, u, v) are coordinates in \mathbf{P}^5 , f is of degree 3 and defines a smooth surface in \mathbf{P}^3 , see [Lat 1, Rk. 18]. For each fourfold $X \in \mathcal{F}$ there is an associated K3 surface S (a double cover of \mathbf{P}^2 ramified along a sextic) and a correspondence $\Gamma \in A^3(S \times X) = \text{End}_{\mathcal{M}_{rat}}(h(S)(1), h(X))$ inducing an isomorphism $\Gamma_* : A_0(S)_0 \simeq A_1(X)_{hom} = A_1(X)_{alg}$, see [Vois 3, 4.2]. By Lemma 4.2 the correspondence Γ induces a map of motives $\gamma : t_2(S)(1) \rightarrow t(X)$ such that $\gamma_* : A^i(t_2(S)(1)) \rightarrow A^i(t(X))$ is an isomorphism for all $i \geq 0$. Therefore, by Lemma 4.3, γ is an isomorphism of motives. The motives of all fourfolds X in \mathcal{F} are finite dimensional and of abelian type, see [Lat 1, Cor.17(iii)], and hence also the motives $t_2(S)$ are finite dimensional and of abelian type. It follows that all the associated K3 surfaces S have a finite dimensional motive of abelian type.

(2) Examples of rational cubic fourfolds X such that the transcendental motive $t(X)$ is isomorphic to $t_2(S)(1)$, for a given K3 surface S can be constructed as in [Has 3, Sect.5]. Let $A^1(S)$ be generated by two ample divisors h_1 and h_2 such that $\langle h_1, h_1 \rangle = \langle h_2, h_2 \rangle = 2$ and $\langle h_1, h_2 \rangle = \langle h_2, h_1 \rangle = 2k + 1$, where $k = 2, 3$ and \langle, \rangle is the intersection form on $H^2(S, \mathbf{Z})$. Then there is diagram

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \subset \mathbf{P}^5 \\ \pi \downarrow & & \\ \mathbf{P}^2 \times \mathbf{P}^2 & & \end{array}$$

where Y is smooth, π is the blow up of a surface S' , the image of $(s_1, s_2) : S \rightarrow \mathbf{P}^2 \times \mathbf{P}^2$ where $s_i : S \rightarrow \mathbf{P}^2$ is the branched double cover given by $\mathcal{O}_S(h_i)$. The map ϕ is obtained by blowing up a plane P and surface T on X . If $k = 2$ then also T is a plane, while T is a Veronese surface in the case $k = 3$. In both cases we have $t_2(P) = t_2(T) = 0$. The motive $h(Y)$ splits as $h(S')(1) \oplus h(\mathbf{P}^2 \times \mathbf{P}^2)$, where $\mathbf{P}^2 \times \mathbf{P}^2$ has no transcendental motive, and $h(X)$ is a direct summand of $h(Y)$. Therefore the transcendental motive $t(X)$ coincides with $t_2(S')(1)$ and $t_2(S') = t_2(S)$, because $t_2(-)$ is a birational invariant.

(3) Let \mathcal{C}_{14} be the irreducible divisor of special cubic fourfolds of degree 14, for which the special surface is a smooth quartic rational normal scroll. By the results in [BR] all the fourfolds X in \mathcal{C}_{14} are rational. Moreover if $X \in (\mathcal{C}_{14} - \mathcal{C}_8)$, then $F(X) \simeq S^{[2]}$, where S is the K3 surface of degree 14 and genus 8 parametrizing smooth quartic

rational normal scrolls contained in X . By Theorem 4.6 there is an isomorphism between the transcendental motives $t(X) \simeq t_2(S)(1)$.

According to Corollary 4.9 if X is a special cubic fourfold with $F(X) \simeq S^{[2]}$, and $h(X)$ is finite dimensional, then $t(X)$ is indecomposable. The following proposition shows that, if X is very general and $h(X)$ is finite dimensional, then $t(X)$ is indecomposable.

Proposition 4.12. *Let X be a very general cubic fourfold. If $h(X)$ is finite dimensional the transcendental motive $t(X)$ is indecomposable.*

Proof. Let the primitive motive $h(X)_{prim} = (X, \pi_{prim}, 0)$ be defined as in [Ki, 8.4], where

$$\pi_{prim} = \Delta_X - (1/3) \sum_{0 \leq i \leq 4} (\gamma^{4-i} \times \gamma^i).$$

and

$$H^*(h(X)_{prim}) = H^4(X, \mathbf{Q})_{prim} = P(X)_{\mathbf{Q}}$$

If X is very general then $\rho_2(X) = 1$ and $A^2(X)$ is generated by the class γ^2 . Therefore in the Chow-Künneth decomposition of $h(X)$ in (4.1) we have $h(X)_{prim} = h_4^{tr}(X) = t(X)$ and

$$h_4(X) = h_4^{alg}(X) + h_4^{tr}(X) \simeq \mathbf{L} \oplus h(X)_{prim}.$$

If X is very general, then $\text{End}_{HS}(H^4(X, \mathbf{Q})_{prim}) = \mathbf{Q}[id]$, see [Vois 2, Lemma 5.1]. Let $\mathcal{M}_{hom}(\mathbf{C})$ be the category of homological motives and let $\tilde{\mathcal{M}}_{hom}(\mathbf{C})$ be the subcategory generated by the motives of all smooth projective varieties V such that the Künneth components of the diagonal in $H^*(V \times V)$ are algebraic. The Hodge realization functor

$$H_{Hodge} : \mathcal{M}_{rat}(\mathbf{C}) \rightarrow HS_{\mathbf{Q}}$$

to the Tannakian category of \mathbf{Q} -Hodge structures induces a faithful functor $\tilde{\mathcal{M}}_{hom}(\mathbf{C}) \rightarrow HS_{\mathbf{Q}}$. Let $\bar{h}(X) = h^{hom}(X) \in \tilde{\mathcal{M}}_{hom}(\mathbf{C})$: then $\text{End}_{\mathcal{M}_{hom}}(\bar{h}(X)_{prim}) \simeq \mathbf{Q}[id]$ and hence

$$\text{End}_{\mathcal{M}_{hom}}(\bar{h}_4^{tr}((X))) \simeq \text{End}_{\mathcal{M}_{hom}}(\bar{h}(X)_{prim}) \simeq \mathbf{Q}[id]$$

If $h(X)$ is finite dimensional then the indecomposability of $\text{End}_{\mathcal{M}_{hom}}(\bar{h}_4^{tr}((X)))$ in $\mathcal{M}_{hom}(\mathbf{C})$ implies the indecomposability in $\mathcal{M}_{rat}(\mathbf{C})$. Therefore

$$\text{End}_{\mathcal{M}_{rat}}(t(X)) \simeq \text{End}_{\mathcal{M}_{rat}}(h(X)_{prim}) \simeq \mathbf{Q}[id]$$

and the transcendental motive of X is indecomposable. □

REFERENCES

- [ABB] A.Auel, C.Böhning and H.C. Graf von Bothmer, *The transcendental lattice of the sextic Fermat surface*, arXiv:1306.6798,[math.AG],July 2013
- [Add] N.Addington, *On two rationality conjectures for cubic four folds*. Math. Res. Letters 23 (1), (2016) , 1-13.
- [Ba] K. Banerjee, *Algebraic cycles on the Fano variety of a cubic fourfold*, arXiv:1609.05627v1[math.AG], Sept 2016.
- [Beau] A.Beauville, *Some surfaces with maximal Picard number*, Journal de l'Ecole polytechnique,Tome I (2014), 101-116
- [BR] M.Bolognesi and F.Russo, *Some loci of rational cubic fourfolds*, arXiv:1504.05863 [math.AG],2016
- [deC-M] A.de Cataldo and L.Migliorini, *The Chow Groups and the Motive of the Hilbert scheme of points on a surface*, Journal of Algebra **251**, (2002), 824-848.
- [Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, Heidelberg-New-York, 1984.
- [Gil] S.Gille, *On Chow motives of surfaces*, J. reine angew. Math. 686 (2014),149-166
- [GG] S.Gorchinskiy and V.Guletskii. *Motives and representability of algebraic cycles on threefolds over a field*, J.Alg. Geometry **21** (2012) 343-373.
- [Gu] V.Guletskii,*Motivic obstruction to rationality of a very general cubic hypersurface in \mathbf{P}^5* . <http://arxiv.org/abs/1605.09434v1>.
- [Has 1] B.Hasset, *Special cubic fourfolds*, Compositio Mathematica, **120** (2000)1-23
- [Has 2] B.Hasset, *Cubic fourfolds, K3 surfaces and rationality questions* , Lecture notes for a 2015 CIME-CIRM summer school.
- [Has 3] B.Hasset,*Some rational cubic fourfolds*, J.Algebraic Geom. 8(10(1999), 103-114.
- [KMP] B. Kahn, J. Murre and C. Pedrini, *On the transcendental part of the motive of a surface*, pp. 143–202 in "Algebraic cycles and Motives Vol II", London Math. Soc. LNS **344**, Cambridge University Press, 2008.
- [Kahn] B. Kahn *Zeta functions and Motives*, Pure and Applied Mathematics Quarterly, Volume 5, Number 1, (2009), 507-570
- [Ki] S.I.Kimura. *A note on 1-dimensional motives* in "Algebraic cycles and Motives Vol II", London Math. Soc. LNS **344**, Cambridge University Press, 2008, 203-213.
- [Ku] V.Kulikov. *A remark on the non rationality problem for generic cubic fourfolds*, Mat. Zametki, Vol.83,No 1(2008) 61-68.
- [Lat] R.Laterveer. *A remark on the motive of the Fano variety of lines of a cubic*, arXiv:1611.08818v1 [math.AG] , November 2016
- [MNP] J.Murre, J.Nagel and C.Peters, *Lectures on the theory of pure Motives*, AMS University Lectures Ser. Vol 61 (2013)
- [SV] M.Shen and C.Vial.*The Fourier transform for certain Hyperkähler fourfolds*, Memoirs of the AMS 240, no. 1139, (2014), 1-104.
- [TZ] Z.Tian and R.Zong, *On cycles on rationally connected varieties*, Compos. Math. 150, no. 3, (2014),396-408.
- [Vial 1] C.Vial. *Niveau and coniveau filtrations on cohomology groups and Chow groups*, Proceedings of LMS 106(2), (2013), 410-444.
- [Vial 2] C.Vial. *Projectors in the intermediate algebraic Jacobians* New York J.Math,19 (2013), 793-822

- [Vial 3] C.Vial *Chow Künneth decomposition for 3-folds and 4-folds fibered by varieties with trivial Chow groups of 0-cycles*, J.of Alg. Geom. 24 (2015),51-80
- [Vial 4] C.Vial. *Algebraic cycles and fibrations*, Doc.Math **18**,(2013), 1521-1553.
- [Vois 1] C.Voisin.*Bloch's conjecture for Catanese and Barlow surfaces*, J. Differential Geometry 97 (2014) 149-175.
- [Vois 2] C.Voisin.*On the universal CH_0 of cubic hyper surfaces*, arXiv:1407.726iv2.
- [Vois 3] C.Voisin.*Remarks on zero-cycles of self-products of varieties*, in: Moduli of vector bundles, Proceedings of the Taniguchi Congress (M. Maruyama, editor), Marcel Dekker New York Basel Hong Kong (1994)
- [Vois 4] *On the Chow ring of certain Hyper Kähler Manifolds*, Pure and Applied Math.Quaterly, volume 4,no. 3 (2008), 1-37.
- [Vois 5] *Hodge Structures , Coniveau and Algebraic Cycles*, in "The Legacy of Bernhard Riemann After one Hundred and Fifty Years". ALM 35 (2014) 719-745.
- [Za] Y.Zarhin . *Algebraic cycles over cubic fourfolds*, Boll.Un.Mat Ital. Vol 7,No4-B(1990),833-847.